KOSZULITY AND THE HILBERT SERIES OF PREPROJECTIVE ALGEBRAS

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1. Introduction

The goal of this paper is to prove that if Q is a connected non-Dynkin quiver then the preprojective algebra $\Pi_Q(k)$ of Q over any field k is Koszul, and has Hilbert series $(1 - Ct + t^2)^{-1}$, where C is the adjacency matrix of the double $\bar{Q}$ of Q.

We also prove a similar result for the partial preprojective algebra $\Pi_{Q,J}(k)$ of any connected quiver Q, where $J \subset I$ is a subset of the set I of vertices of Q. By definition, $\Pi_{Q,J}(k)$ is the quotient of the path algebra of $k\bar{Q}$ by the preprojective algebra relations imposed only at vertices not contained in J. We show that if $J \neq \emptyset$ then $\Pi_{Q,J}(k)$ is Koszul, and its Hilbert series is $(1 - Ct + D_J t^2)^{-1}$, where D_J is the diagonal matrix with $(D_J)_{ii} = 0$ if $i \in J$ and $(D_J)_{ii} = 1$, $i \notin J$.

Moreover, we show that both results are valid in a slightly more general framework of modified preprojective algebras, considered in [K].

We note that the first result is known in most cases [MV, MOV, O]. In particular, it is known in general in characteristic zero ([MOV]), and in most positive characteristic cases [MV, O]. Our argument, however, is elementary, and different from the arguments of [MOV, O], which are based on the theory of tensor categories.

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2. Preliminaries

In this section we give some basic definitions and known results, which will be useful in the sequel.

2.1. **Hilbert series of bimodules.** Let k be a field (of any characteristic). Let I be a finite set, and let $R = \oplus_{i \in I} k$ be the algebra of k-valued functions on I. An R-bimodule W may be thought of as an $I \times I$-graded vector space $W = \oplus_{i,j \in I} W_{i,j}$. Then for two R-bimodules U, W we have

$$\left(W \underset{R}{\otimes} U \right)_{i,j} = \bigoplus_{s \in I} W_{i,s} \underset{k}{\otimes} U_{s,j}. \tag{2.1.1}$$

The *tensor algebra* $T_R(W)$ is defined as $\oplus_{m\geq 0} W^{\underset{R}{\otimes} m}$, with the tensor products over R.

Now let $W = \oplus_{d\geq 0} W[d]$ be a $\mathbb{Z}_+$-graded R-bimodule, with finite dimensional homogeneous subspaces.

Definition 2.1.2. *We define the* Hilbert series $h_W(t)$ *to be a matrix-valued series, with the entries*

$$h_W(t)_{i,j} = \sum_{d=0}^{\infty} dim\ W[d]_{i,j} t^d.$$

From (2.1.1), it follows that $h_{W\underset{R}{\otimes}U}(t) = h_W(t)h_U(t)$.

2.2. **Free products.** Let A, B be k-algebras containing R.

Definition 2.2.1. *We define the* free product $A \underset{R}{*} B$ *of A and B over R to be the algebra $T_R(A \oplus B)/J$, where J is the ideal generated by elements*

1) $r_A - r_B$, where $r \in R$ and r_A, r_B are its images in A and B, respectively, and

2) $a_1 \otimes a_2 - a_1a_2$, $a_1, a_2 \in A$; $b_1 \otimes b_2 - b_1b_2$, $b_1, b_2 \in B$.

Example 2.2.2. *If $A = T_R(V)/(r_i)$, $B = T_R(W)/(s_j)$ and r_i, s_j are the defining relations, then $A \underset{R}{*} B = T_R(V \oplus W)/(r_i,\, s_j)$.*

Example 2.2.3. *Assume that we have decompositions of bimodules $A = R \oplus A_+$, $B = R \oplus B_+$. Then*

$$A \underset{R}{*} B = \bigoplus_{m=0}^{\infty} A \underset{R}{\otimes} \left(B_+ \underset{R}{\otimes} A_+\right)^m \underset{R}{\otimes} B \tag{2.2.4}$$

Lemma 2.2.5. *Suppose A, B are $\mathbb{Z}_+$-graded algebras, with $A[0] = B[0] = R$. Let $h_A(t) = \frac{1}{1-\alpha}$, $h_B(t) = \frac{1}{1-\beta}$. Then $h_{A\underset{R}{*}B}(t) = \frac{1}{1-\alpha-\beta}$.*

Proof. Let A_+, B_+ be the positive degree parts of A, B. Then we have

$$h_{A_+}(t) = \frac{\alpha}{1-\alpha}, h_{B_+}(t) = \frac{\beta}{1-\beta},$$

$$h_{\oplus_{m\geq 0}\left(B_+\underset{R}{\otimes}A_+\right)^m}(t) = \sum_{m=0}^{\infty}\left(\frac{\beta}{1-\beta}\frac{\alpha}{1-\alpha}\right)^m = \left(1 - \frac{\beta}{1-\beta}\frac{\alpha}{1-\alpha}\right)^{-1}.$$

By (2.2.4), we have

$$\begin{aligned} h_{A\underset{R}{*}B}(t) &= (1-\alpha)^{-1}\left(1 - \frac{\beta}{1-\beta}\frac{\alpha}{1-\alpha}\right)^{-1}(1-\beta)^{-1} \\ &= ((1-\beta)(1-\alpha) - \beta\alpha)^{-1} = \frac{1}{1-\alpha-\beta}. \end{aligned}$$

□

2.3. **Quadratic and Koszul algebras.** For the material in this subsection, we refer to [PP].

Let V be an R-bimodule, and $E \subset V \underset{R}{\otimes} V$ an R-subbimodule. To this data we may attach the *quadratic algebra* $A := T_R(V)/(E)$.

Definition 2.3.1. *Assume that the bimodule V is equipped with a filtration by nonnegative integers. Then we define $A' := T_R(\mathrm{gr}V)/(\mathrm{gr}E)$.*

The following lemma is obvious.

Lemma 2.3.2. *We have the termwise inequality $h_{A'}(t) \geq h_A(t)$. In other words, for every $i, j \in A$ and $d \geq 0$ we have $\dim A'[d]_{i,j} \geq \dim A[d]_{i,j}$.*

Definition 2.3.3. *Let A be a quadratic algebra with $A[0] = R$. A is a* Koszul algebra *if $Ext^i_A(R,R)$ (or dually $Tor^A_i(R,R)$) is concentrated in degree i.*

Theorem 2.3.4. *(The Golod-Shafarevich inequality) Let A be a $\mathbb{Z}_+$-graded algebra with $A[0] = R$, with the bimodule of generators V in degree 1 and relations E in degree 2. Let C and D be the matrices defined by $C_{i,j} = \dim V_{i,j}$, $D_{i,j} = \dim E_{i,j}$. Then:*
(i) If $\frac{1}{1-Ct+Dt^2} \geq 0$ termwise, then $h_A(t) \geq \frac{1}{1-Ct+Dt^2}$.
(ii) If $h_A(t) = \frac{1}{1-Ct+Dt^2}$, then A is Koszul.

Proof. From the exact Koszul complex,

$$0 \to K \to A \underset{R}{\otimes} E \to A \underset{R}{\otimes} V \to A \to R \to 0$$

extended by the kernel K on the left, using the Euler-Poincaré principle, we obtain the equation

$$h_A(t)(1 - Ct + Dt^2) - 1 = h_K(t) \geq 0,$$

so under the assumption of (i) the inequality $h_A(t) - \frac{1}{1-Ct+Dt^2} \geq 0$ follows. To prove (ii) note that in the above exact sequence, $h_A(t) = \frac{1}{1-Ct+Dt^2}$ implies $h_K(t) = 0$, therefore $K = 0$. Applying the functor $R \underset{A}{\otimes} \_$ to this sequence (without the last term R), we get the complex computing $Tor^A_i(R,R)$:

$$0 \xrightarrow{0} E \xrightarrow{0} V \xrightarrow{0} R \xrightarrow{0} 0.$$

So

$$Tor^A_i(R,R) = \begin{cases} R & i = 0 \\ V & i = 1 \\ E & i = 2 \\ 0 & i > 2, \end{cases}$$

and hence A is Koszul. $\square$

3. Hilbert series of preprojective algebras

3.1. **Partial preprojective algebras.** Let Q be a finite quiver with vertex set I (we allow multiple edges and self-loops). For each $a \in Q$ let $h(a), t(a) \in I$ denote the head and tail of the edge a. Let V be the k-vector space spanned by the edges of Q; it is naturally an R-bimodule, where $R = \oplus_{i\in I} k$. Recall that the path algebra kQ is the algebra $T_R V$.

Let $\bar{Q}$ the *double* of Q, obtained from Q by keeping the same vertex set and adding a new edge a^* for j to i for each edge a from i to j.

Let $J \subset I$ be a subset of "white" vertices (the other vertices are colored black).

Definition 3.1.1. *We define the* partial preprojective algebra

$$\Pi_{Q,J}(k) = k\bar{Q} \Big/ \left(\sum_{a\in Q, h(a)\in I\setminus J} aa^* - \sum_{a \in Q, t(a)\in I\setminus J} a^* a \right)$$

In particular, if $J = \emptyset$ then we write $\Pi_Q(k)$ for $\Pi_{Q,\emptyset}(k)$ and call it the preprojective algebra *of Q.*

Remark 3.1.2. In a similar way, for any commutative ring K one defines the algebra $\Pi_{Q,J}(K)$. When no confusion is possible, we will denote $\Pi_{Q,J}(K)$ by $\Pi_{Q,J}$.

3.2. **The Hilbert series for an extended Dynkin quiver.** Let Q be an extended Dynkin quiver (in particular, we allow the quiver $\widetilde{A}_0$, which has one vertex and one edge).

Proposition 3.2.1. *The Hilbert series $h_{\Pi_Q(k)}(t)$ equals $\frac{1}{1-Ct+t^2}$, where C is the adjacency matrix of $\bar{Q}$.*

Proof. First we show this for $\mathrm{char} k = 0$. (In this case the result is well known, and the argument actually works if p is a good prime for Q, in particular for any $p > 5$).

Let $\Gamma \subset SL_2(k)$ be the finite group attached to Q via the McKay's correspondence, f_i the primitive idempotents of the irreducible representations of Γ, $f = \sum_i f_i$, and $A = \Gamma \ltimes k[x,y]$ the skew group algebra.

Then we have the following result, due to G. Lusztig [L] (see also [CBH]).

Proposition 3.2.2.

$$\Pi_Q(k) = \bigoplus_{i,j} f_i A f_j = fAf$$

By applying the functor $\_ \otimes k\Gamma$ to the exact Koszul complex of $k[x,y]$ (over k) we obtain the exact complex $\widetilde{K}^\bullet$:

$$0 \longrightarrow A(2) \longrightarrow A \underset{k\Gamma}{\otimes} A[1] \longrightarrow A \longrightarrow k\Gamma \longrightarrow 0$$

(here (2) denotes the shift in grading by 2). By Proposition 3.2.2, the complex $K^\bullet = f\tilde{K}^\bullet f$ has the form

$$0 \longrightarrow \Pi_Q(k)(2) \longrightarrow \Pi_Q(k) \underset{R}{\otimes} \Pi_Q(k)[1] \longrightarrow \Pi_Q(k) \longrightarrow R \longrightarrow 0.$$

We conclude that this complex is exact, and from the Euler-Poincaré principle obtain the equation

$$1 = h_R(t) = h_{\Pi_Q(k)}(t) - h_{\Pi_Q(k)}(t)Ct + h_{\Pi_Q(k)}(t)t^2.$$

Now consider the case char$k \neq 0$. Let T be the torsion part of $\Pi_Q(\mathbb{Z})$. Then $\Pi_Q(\mathbb{Z})/T$ is a free $\mathbb{Z}$-module, and by the characteristic zero result has the Hilbert series $\frac{1}{1-Ct+t^2}$ (indeed, it suffices to take tensor product with $\mathbb{C}$). Further, $T \underset{\mathbb{Z}}{\otimes} k \subset \Pi_Q(k)$ is an ideal, and the quotient algebra $\Pi_Q(k)/T \underset{\mathbb{Z}}{\otimes} k = (\Pi_Q(\mathbb{Z})/T) \underset{\mathbb{Z}}{\otimes} k$ has the same Hilbert series. Since C is an extended Cartan matrix, its largest eigenvalue is 2 and hence we get that $\Pi_Q(k)/T \underset{\mathbb{Z}}{\otimes} k$ has Gelfand-Kirillov dimension 2. By [BGL], $\Pi_Q(k)$ is a prime Noetherian algebra of Gelfand-Kirillov dimension 2. But for any prime Noetherian algebra A of Gelfand-Kirillov dimension d and any nonzero two-sided ideal $I \subset A$, one has $\mathrm{GKdim}(A/I) \leq d-1$, [MR, Corollary 8.3.6.]. Therefore, we see that $T \underset{\mathbb{Z}}{\otimes} k = 0$, and hence $T = 0$, as desired. □

3.3. **Star-shaped quivers.** In this subsection we prove the Hilbert series formula for star-shaped quivers with node being a white vertex.

Lemma 3.3.1. *Let Q be a quiver with vertex set $I = \{1, ..., n+1\}$ and arrows $n+1 \xrightarrow{a_{i,1},\ldots a_{i,r_i}} i \in I$, and with a set of white vertices $J = \{i_1, \ldots i_m, n+1\}$. Then $h_{\Pi_{Q,J}}(t) = \frac{1}{1-Ct+D_Jt^2}$ where D_J is a diagonal matrix with $(D_J)_{ii} = \begin{cases} 1 & i \notin J \\ 0 & i \in J. \end{cases}$*

Proof. First, we observe that $\Pi_{Q,J} = A_1 * \ldots * A_{n+1}$ where A_i is the partial preprojective algebra of the quiver Q_i with vertex set I and only the arrows $n+1 \xrightarrow{a_{i,1},\ldots a_{i,r_i}} i$ (if $i = n+1$, these arrows are self-loops). By Lemma 2.2.5, it is enough to prove the result in the case when this free product has only one factor, i.e. for $n = 1$, and for $n = 2$, $r_2 = 0$.

If all vertices are white, then $\Pi_{Q,J}$ is a path algebra and the result is clear from counting paths. So it remains to consider the case $n = 2$, $r_2 = 0$, where 1 is a black vertex. In this case, the quiver Q has edges $a_1, ..., a_r$ going from 2 to 1, and we have one defining relation

$$a_1a_1^* + ... + a_ra_r^* = 0.$$

Denote the algebra $\Pi_{Q,J}$ in this case by $A(r)$.

Consider first the case $r = 1$. In this case the algebra $A(r)$ has only one quadratic element $a_1^*a_1$ (up to scaling) and no cubic elements, so the formula easily follows.

Now consider the case $r > 1$. Let us introduce the filtration on $A := A(r)$ by setting $\deg(a_1) = \deg(a_1^*) = 1$, and $\deg(a_i) = \deg(a_i^*) = 0$ for $i > 1$. In this case $A' = A(1) \underset{R}{*} B$, where B is the path algebra of the quiver with edges a_i, a_i^*, $i = 2, ..., r$. It then follows from Lemma 2.2.5 and the $r = 1$ case that the desired Hilbert series formula holds for A'. By Lemma 2.3.2, this implies that

$$h_A(t) \leq \frac{1}{1 - Ct + D_J t^2}.$$

Then by Theorem 2.3.4, we see that

$$h_A(t) = \frac{1}{1 - Ct + D_J t^2},$$

as desired. □

3.4. **Main results.**

Theorem 3.4.1. *Let Q be a connected quiver and a nonempty set of white vertices. Then $h_{\Pi_{Q,J}}(t) = \frac{1}{1-Ct+D_J t^2}$. In particular, $\Pi_{Q,J}$ is a Koszul algebra.*

Proof. The second statement follows from the first statement and Theorem 2.3.4. So we only need to prove the first statement.

We will prove by induction in the number of vertices that the statement is true for any quiver Q whose every connected component contains a white vertex.

If Q has just one vertex, the statement is clear.

To make the induction step, assume the formula is right for $\leq n$ vertices, and that Q has $n + 1$ vertices.

Let I be the vertex set of Q. Select any white vertex w in I and consider the subquiver Q_0 with vertex set $I_0 \subset I$, consisting of w and the vertices adjacent to it, whose arrows are all the arrows of Q which touch w. Also, let Q' be the quiver with the vertex set $I \setminus \{w\}$, where the vertices in I_0 are colored white and the other ones are colored in the same way as in Q, and with the set of arrows $Q \setminus Q_0$. Finally, let $\widehat{Q}_0$, $\widehat{Q}'$ be the quivers with vertex set I and with the same arrows as in Q_0, Q' respectively, and let J_0, J' be the sets of white vertices of Q_0 and Q'.

Introduce a filtration on $\Pi_{Q,J}$, by setting a grading, such that the arrows inside $\bar{Q}_0$ have degree 1 and the other ones have degree 0. Then $\Pi'_{Q,J} = \Pi_{\widehat{Q}_0,J_0} * \Pi_{\widehat{Q}',J'}$. Therefore, by Lemma 2.3.2, we get

$$h_{\Pi_{Q,J}}(t) \leq h_{\Pi'_{Q,J}}(t) = h_{\Pi_{\widehat{Q}_0,J_0} * \Pi_{\widehat{Q}',J'}}(t).$$

Now, by Lemma 3.3.1 for Q_0, $h_{\Pi_{\widehat{Q_0},J_0}}(t) = \frac{1}{1-C_0 t+D_0 t^2}$ where C_0 is the adjacency matrix of $\bar{Q}_0$, and D_0 is the diagonal matrix, such that $(D_0)_{ii} = 1$ if i is a black vertex in I_0, and 0 otherwise. Also, applying the induction assumption to Q', we find $h_{\Pi_{\widehat{Q'},J'}}(t) = \frac{1}{1-(C-C_0)t+D' t^2}$, where D' is the diagonal matrix with $(D')_{ii} = 1$ if i is a black vertex $\notin I_0$, and 0 otherwise.

Therefore, by Lemma 2.2.5 we have $h_{\Pi_{Q,J}}(t) \leq \frac{1}{1-Ct+D_J t^2}$. From this, it follows $\frac{1}{1-Ct+D_J t^2} \geq 0$. Now, by Theorem 2.3.4, the result follows. $\square$

Theorem 3.4.2. *The Hilbert series $h_{\Pi_Q}(t)$ for a connected non-Dynkin quiver Q is $\frac{1}{1-Ct+t^2}$. In particular, Π_Q is Koszul.*

Proof. Again, the second statement follows from the first one and Theorem 2.3.4, and we only prove the first statement.

For this, we use the following easy (and well known) lemma.

Lemma 3.4.3. *Any connected non-Dynkin quiver Q contains an extended Dynkin subquiver Q_E.*

Let I_E be the vertex set of Q_E (it is possible that an arrow a between $i, j \in I_E$ belongs to Q but not to Q_E). Let Q' be the quiver with vertex set I, such that the vertices in I_E are white and the other ones are black, and the set of arrows $Q \setminus Q_E$. Then every connected component of Q' contains at least one white vertex. Introduce a filtration on Π_Q by setting a grading, such that the arrows in $\bar{Q}_E$ have degree 1 and the other ones have degree 0. Then $\Pi'_Q = \Pi_{\widehat{Q}_E} * \Pi_{Q',I_E}$, where $\widehat{Q}_E$ is the quiver Q_E with adjoined vertices of $I \setminus I_E$.

Now, by Proposition 3.2.1, $h_{\Pi_{\widehat{Q}_E}} = \frac{1}{1-C_E t+D_E t^2}$ where D_E is the diagonal matrix, such that $(D_E)_{ii}$ is 1 if $i \in I_E$ and 0 otherwise, and C_E is the adjacency matrix of of the double of $\widehat{Q}_E$. Also, by Theorem 3.4.1, $h_{\Pi_{Q',I_E}} = \frac{1}{1-(C-C_E)t+(1-D_E)t^2}$. Hence, by Lemma 2.2.5, we obtain

$$h_{\Pi_Q} \leq h_{\Pi'_Q} = h_{\Pi_{\widehat{Q}_E} * \Pi_{Q',I_E}} = \frac{1}{1-Ct+t^2}.$$

From this, it follows that $\frac{1}{1-Ct+t^2} \geq 0$. Hence, Theorem 2.3.4 implies the result. $\square$

3.5. **Modified preprojective algebras.** It turns out that our results hold for a slightly more general class of preprojective algebras. Namely, let Q be a quiver with vertex set I, and $\bar{Q}$ its double. Let $J \subset I$. Let γ be a $k^\times$-valued function on the set of edges of $\bar{Q}$ which begin or end at $I \setminus J$. Define the modified partial preprojective algebra $\Pi^\gamma_{Q,J}(k)$ to be the quotient of $k\bar{Q}$ by the relation

$$\sum_{a \in Q, h(a) \in I \setminus J} \gamma_a a a^* - \sum_{a \in Q, t(a) \in I \setminus J} \gamma_{a^*} a^* a = 0.$$

Obviously, this is a generalization of the usual partial preprojective algebras $\Pi_{Q,J}(k)$, which are obtained if $\gamma = 1$.

If $J = \emptyset$, $\Pi^\gamma_{Q,J}(k)$ is called a modified preprojective algebra and denoted by $\Pi^\gamma_Q(k)$. Such algebras are considered in [K].

Theorem 3.5.1. *Theorem 3.4.1 and Theorem 3.4.2 hold for the algebras $\Pi^\gamma_{Q,J}(k)$ and $\Pi^\gamma_Q(k)$.*

Proof. The proof of Theorem 3.4.1 carries out verbatim to the case of general γ.

To prove Theorem 3.4.2 for Π_Q^γ, we only need to consider the extended Dynkin case, since it is the only case when the proof needs to be changed. Also, note that if Q is a tree, then the algebras Π_Q^γ are pairwise isomorphic for all choices of γ (by rescaling the edges). So it is necessary to consider only the case of type $\widetilde{A}_{n-1}$.

In this case the edges are $a_1, ..., a_n$ and $a_1^*, ..., a_n^*$ (where a_i goes from i to $i+1$), and the defining relations are

$$\gamma_{a_i^*} a_i^* a_i - \gamma_{a_{i-1}} a_{i-1} a_{i-1}^* = 0,$$

where $i-1$ is computed modulo n.

These relations show that Π_Q^γ is spanned by paths in which there is no expressions $a_i^* a_i$. Counting such paths, we easily get that

$$h_{\Pi_Q^\gamma} \leq (1 - Ct + t^2)^{-1},$$

which by Theorem 2.3.4 implies the result. □

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